# No-Signalling Is Equivalent To Free Choice Of Measurements

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We shall also sketch an 'operational interpretation' of negative probabilities.

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  So the underlying structural reasons **why** it holds seem to be different.

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Apparently of a very different character to No-Signalling!

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Thus $(U, s)$ is the event that the measurements in $U$ were performed, and the outcome $s(x)$ was observed for each $x \in U$.

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Also generalises K-S configurations: given a family of observables $X$, take $\mathcal{U}$ to the the family of commuting subsets.

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We say that an schv model $m$ **realizes** an empirical model $e$ if $\hat{m} = e$.

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The distribution on the hidden variables should be statistically independent of the choice of measurement context.

Formally, an schv model $m$ satisfies FCM if it factors as a product: $m = m_{\Omega} m_{\mathcal{U}}$, where $m_{\Omega} \in \mathsf{D}^{\pm}(\Omega)$ and $m_{\mathcal{U}} \in \mathsf{D}^{\pm}(\mathcal{U})$ are the marginals of $m$.

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*An schv model m satisfies FCM iff for all* $\omega \in \Omega$, $U, U' \in \mathcal{U}$,

$$m(\omega|U) = m(\omega|U').$$

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**Theorem (Bell's Theorem)**

*There are empirical models which can be realized in quantum mechanics which do not admit local hidden variables.*

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Note however that the converse does not hold, even for pchv models.

# The Main Result

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**Theorem**

*Empirical models have local hidden-variable realizations with negative probabilities if and only if they satisfy no-signalling.*

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*The linear subspace generated by the local models over an arbitrary measurement cover $\mathcal{U}$ coincides with that generated by the no-signalling models. Their common dimension is*

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Since the local models are included in the no-signalling models, this is proved by showing that every compatible model is determined by linear equations in $D$ variables; while there are $D$ linearly independent local models.

# Example: PR Boxes have global sections over $\mathbb{R}$

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The 'Popescu-Rohrlich box':

| | $(0,0)$ | $(1,0)$ | $(0,1)$ | $(1,1)$ |
|---|---|---|---|---|
| $(a,b)$ | $1/2$ | $0$ | $0$ | $1/2$ |
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This vector can be taken as giving a **local hidden-variable realization of the PR box using negative probabilities**. Similar explicit realizations can be given for the other PR boxes.

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Hence we obtain the equivalence:

## Theorem

*An empirical model is No-Signalling if and only if it is realized by an schv model satisfying FCM.*

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The signed relative frequency of an event $\bullet$ in an ensemble of size $N$ will be

$$\frac{n^{+} - n^{-}}{N}$$

where $\bullet^{+}$ occurs $n^{+}$ times and $\bullet^{-}$ occurs $n^{-}$ times in the ensemble.

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$$\theta_X : \mathsf{D}^{\pm}(X) \longrightarrow \mathsf{D}^{+}(X + X)$$

which puts positive weight $d(x) \geq 0$ on $x$ on positive copy (first summand) and (absolute value of) negative weight on second copy.

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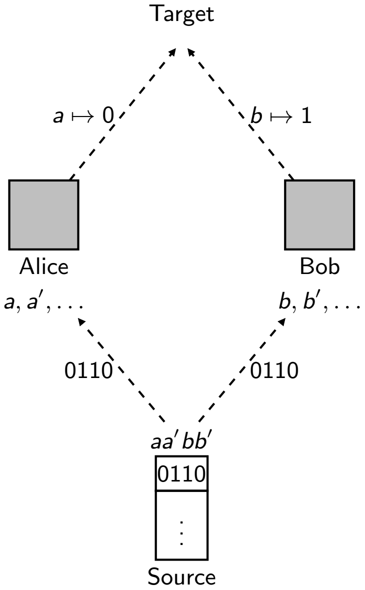
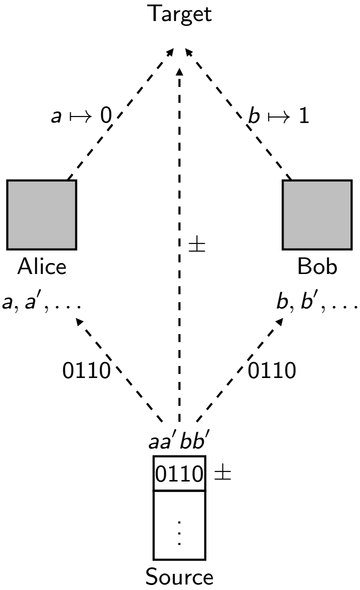
  This is a (pale) reflection of the Hahn-Jordan decomposition.

# The Mermin instruction set picture



# Signed version of the Mermin instruction set picture



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- Uses?
- Does ‘cancellation’ imply or otherwise relate to **retrocausality**?