A Unified Sheaf-Theoretic Account of Non-Locality and Contextuality: Part I

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Hidden Variables

Both passwords are either p2s4w6r8 or 1a3s5o7d.

If Alice types in p2s4w6r8 and this unlocks her computer, then we know what will happen when Bob types in a password.

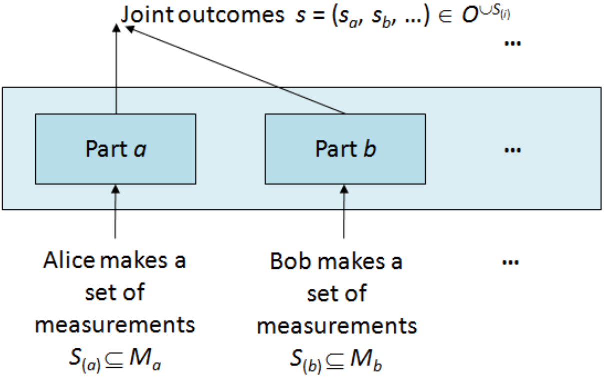
It is our state of knowledge that changes, not Bob's computer.

We can consider an r.v. X (resp. Y) for Alice's (resp. Bob's) password, and an extra r.v. Z taking values z_1 or z_2.

Then X and Y are conditionally independent given Z.

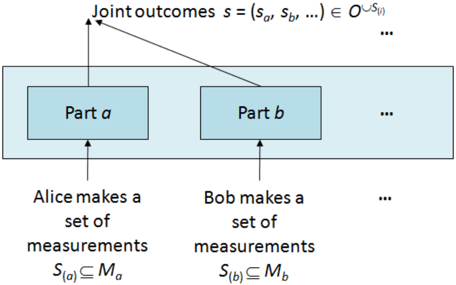
Source: Wikimedia Commons

Experiments



Associated with each part i is a set M_i of **basic measurements**.
Associated with each measurement is a set O of possible **outcomes**.
Take the M_i to be disjoint and let $M = \bigsqcup_i M_i$.

Compatibility Structures

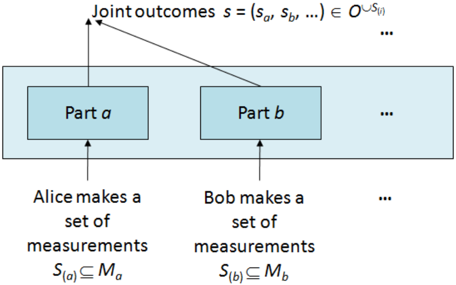


At each part i there is a family $\mathcal{C}_i$ of subsets of M_i, specifying the **compatible** sets of measurements.
This yields a **compatibility structure** $\mathcal{C}$ on M:

$$\mathcal{C} = \{S \subseteq M : S_{(i)} \in \mathcal{C}_i \text{ for all } i\},$$

where $S_{(i)} = S \cap M_i$.

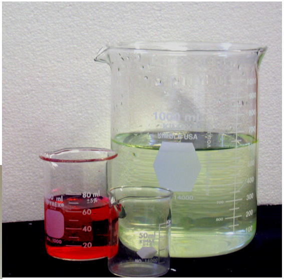
Two Special Cases



Bell locality: Each $\mathcal{C}_i$ consists of the singletons $\{m_i\}$.

Kochen-Specker contextuality: One part where $\mathcal{C}$ consists of more than the singletons but less than the power set.

Modeling the Classical World

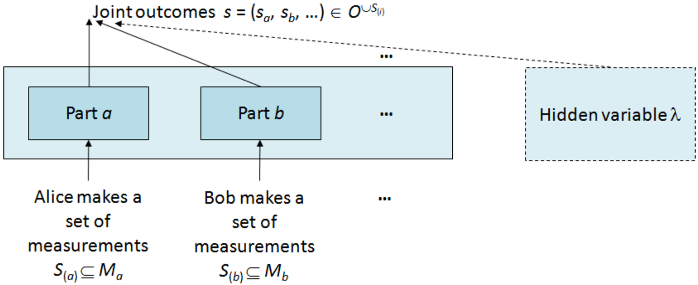


The same framework could be used to analyze classical incompatibilities (if we wish to assume such).

Example: Can we measure both the volume of the chocolate and the flavor of the filling?

Source: Wikimedia Commons

Empirical Models from Hidden-Variable Models



The hidden variable (h.v.) λ interacts with the system to determine (probabilistically) the outcomes of measurements.

H.v. models **realize** empirical models by averaging over values of λ.

We can ask for properties of h.v. models — various forms of (conditional) independence, determinism, ... — that it would be unreasonable to ask of empirical models.

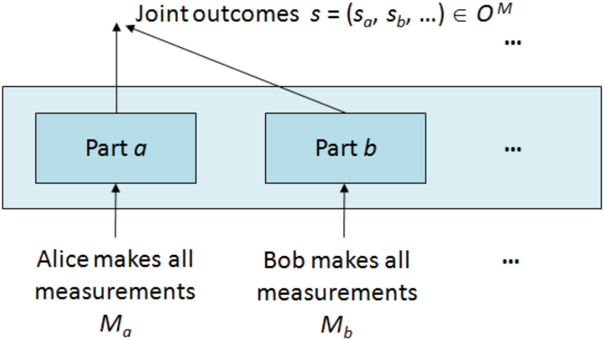
Properties of Empirical Models

Part II of the talk will treat h.v. models.

Here, we will see how far we can go by imposing — reasonable! — properties on empirical models.

Let's begin with a thought experiment . . .

A Tale of Two Laboratories



Suppose that, in another laboratory, Alice (resp. Bob) undertakes *all* her (resp. his) measurements.

A Definition of Classicality

Definition

If there is such a second laboratory — where the empirical frequencies of outcomes agree with those in the first laboratory — we say that the original experiment is classical. Formally, we will call this the **extendability** property of an empirical model.

A Non-Classical Experiment

$M_a = \{0, 2\pi/3\}$, $M_b = \{0, 4\pi/3\}$, $O = \{+, -\}$.
$\mathcal{C}_a = \{\{0\}, \{2\pi/3\}\}$, $\mathcal{C}_b = \{\{0\}, \{4\pi/3\}\}$.

	+ +	− +	+ −	− −
0 0	*	*	1/2	*
2π/3 0	*	*	1/8	*
0 4π/3	*	*	1/8	*
2π/3 4π/3	*	1/8	*	*

Does extendability hold for the Bell empirical model?

The Bell Model cont'd.

	0	$2\pi/3$	0	$4\pi/3$
s^1	+	+	+	+
s^2	–	+	+	+
s^3	+	–	+	+
s^4	+	+	–	+
s^5	+	+	+	–
s^6	–	–	+	+
s^7	+	–	–	+
s^8	+	+	–	–
s^9	–	+	+	–
s^{10}	–	+	–	+
s^{11}	+	–	+	–
s^{12}	–	–	–	+
s^{13}	+	–	–	–
s^{14}	–	+	–	–
s^{15}	–	–	+	–
s^{16}	–	–	–	–

The Bell Model cont'd

	+ +	− +	+ −	− −
0 0	*	*	1/2	*
2π/3 0	*	*	1/8	*
0 4π/3	*	*	1/8	*
2π/3 4π/3	*	1/8	*	*

Extendability requires:

$$\begin{aligned} p(s^4) + p(s^7) + p(s^8) + p(s^{13}) &= 1/2, \\ p(s^4) + p(s^8) + p(s^{10}) + p(s^{14}) &= 1/8, \\ p(s^5) + p(s^8) + p(s^{11}) + p(s^{13}) &= 1/8, \\ p(s^3) + p(s^6) + p(s^7) + p(s^{12}) &= 1/8, \end{aligned}$$

which is impossible!

The E = F Theorem

The usual impossibility argument shows that there is no h.v. model satisfying **locality** and λ-**independence** that realizes the Bell empirical model.

Part II will present a general independence property of h.v. models which we call **factorizability**.

This specializes to **locality** and (a strong form of) **non-contextuality**.

Theorem (E = F)

*An empirical model is extendable iff it can be realized by a factorizable h.v. model.**

* λ-independence is a framework property in our treatment.

The Literature

Fine (*Physical Review Letters*, 1982) appears to have been the first to link incompatibility and non-locality.

Oppenheim and Wehner (*Science*, 11/19/10) relate (entropic) uncertainty relations and non-locality.

See also Liang, Spekkens, and Wiseman (arXiv 1010.1273, 10/06/10).

Our E = F Theorem establishes, at a high level of generality, the fundamental role of incompatibility.

What Would a Probabilist Do?

		+ +	− +	+ −	− −
0	0	0	>0	1/2	0
2π/3	0	>0	>0	1/8	>0
0	4π/3	>0	>0	1/8	>0
2π/3	4π/3	>0	1/8	>0	>0

If extendability fails, it is natural for a probabilist to ask if **absolute continuity** holds.

Can we find a probability measure on the extended empirical model that, for each set of measurements, is absolutely continuous with respect to the given probability measure (on the actual empirical model)?

Absolute Continuity

	0	$2\pi/3$	0	$4\pi/3$
s^1	+	+	+	+
s^2	–	+	+	+
s^3	+	–	+	+
s^4	+	+	–	+
s^5	+	+	+	–
s^6	–	–	+	+
s^7	+	–	–	+
s^8	+	+	–	–
s^9	–	+	+	–
s^{10}	–	+	–	+
s^{11}	+	–	+	–
s^{12}	–	–	–	+
s^{13}	+	–	–	–
s^{14}	–	+	–	–
s^{15}	–	–	+	–
s^{16}	–	–	–	–

Put probability 1 on the indicated row.

Global Sections

Definition

An empirical model has a **global section** if there is a map $s : M \longrightarrow O$ s.t. for each set of compatible measurements $S \in \mathcal{C}$, the restriction of s to S lies in the support of the probability measure given S.

Lemma

An empirical model has a global section iff it satisfies absolute continuity.

(Of course, we can also formulate an equivalence in terms of Radon-Nikodym derivatives. We do not know if this will be useful.)

Strong Contextuality

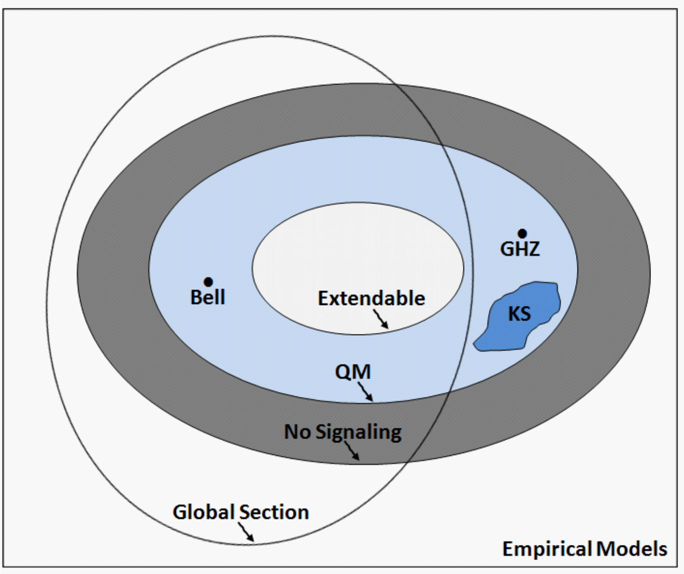
Definition

If empirical model has no global section, we say it is **strongly contextual**.

Part II will formulate a generalized version of the Kochen-Specker theorem, which establishes strong contextuality of a whole *class* of empirical models.

There will also be a model-dependent impossibility result (guess which model!).

Properties of Empirical Models



Some Questions

H.v. characterization of extendability: Our E = F Theorem

H.v. characterization of no signaling: λ-independence and parameter independence (Brandenburger and Keisler 2011)

H.v. characterization of global section: ?